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JOINTS IN OPTIMUM FRAMEWORKS

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Abstract—An objection to the Michell theory of optimum frameworks is that many of the forms achieved embody large numbers of joints, and the theory ignores the penalty in material or fabrication cost which these entail. The paper investigates the forms of the optimum frameworks for some simple load systems when the cost of joints is taken into account.

NOTATION

A	area of triangle
a_1, a_2, a_3	areas of sub-triangles as fractions of A
i, k	suffices taking values 1, 2, 3
j	joint radius
l	length of member
l_1, l_2, l_3	lengths of members of tripedal framework
P	external load
p, q	perpendicular distances defined in Fig. 2
R	force in member
s	span of beam
s_1, s_2, s_3	lengths of members of triangular framework
V	volume of material
α, β	angles defined in Figs. 3 and 4
σ	permissible stress

1. INTRODUCTION

Optimum structures are defined for our present purposes as those which react a given set of forces with a minimum volume of material. The theory of optimum frameworks was first developed by Maxwell [1] and Michell [2]. It has been extended to other types of structure and to multiple loading systems by Foulkes [3], Hemp [4] and others. An earlier discussion of a particular application of the concept will be found in Oliver Wendell Holmes [5].

One of the objections which is commonly raised to the practical application of optimum frameworks is that many of the designs achieved embody very large numbers of joints, and the cost (in material or money) of fabricating these joints would far exceed the saving in bar material relative to a simpler but non-optimal structure. It is the purpose of the present paper to examine the influence of joint cost on the forms of the optimum frameworks which react certain simple but important load systems.

It is first necessary to define the material penalty involved in joining bars of the framework together or in reacting the external forces. In an optimum structure all members are subjected to the maximum permissible stress σ which we shall assume to be the same in tension and compression. We shall further assume that the material (or financial) cost of transferring a force from a member to a gusset plate or other type of connection is proportional to the force transferred: a valuable discussion of whether a linear relation or one based upon dimensional similarity should be used is given in Cox [6]. Since all members are subjected to the same stress, the force in a member is proportional to its cross-sectional area. It follows that our assumption is equivalent to adding to each member at a joint a constant length j, which we shall denote the *joint radius*. The concept is shown in Fig. 1: any external force at the joint is connected through a member of appropriate area and length j.

The value to be assigned to j will vary with the type of construction. For temporary joints j may be quite large. It will be smaller for permanent bolted or riveted joints, smaller still for welded joints and possibly least for some types of reinforced concrete frameworks. The value of j will also depend on whether one is concerned with material consumption or the financial cost of fabrication.

It will be seen that in a framework consisting of bars of length l carrying forces R and reacting external loads P the total volume V of material consumed is $(1/\sigma)\Sigma R(l+2j)+(1/\sigma)\Sigma Pj$.

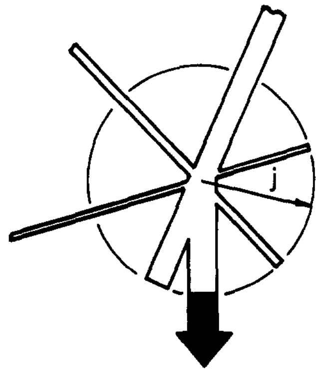

Fig. 1.

The second term is constant for a given load system, so that we are concerned with minimising $\Sigma R(l+2j)$, instead of the usual Maxwell–Michell expression ΣRl. The paper examines this problem in relation to single-sign frameworks for the general three-load system and the full-space and half-space frameworks for a beam carrying a central load.

2. SINGLE-SIGN FRAMEWORKS TO REACT THREE FORCES

We consider an equilibrium set of three forces P_1, P_2 and P_3 applied at points D, E and F as shown in Fig. 2. Then if the point of concurrence C of the forces lies within the triangle DEF they can be reacted by a framework in which all of the bar forces have the same sign. There are an unlimited number of such frameworks and we know from Maxwell[1] that they each employ the same volume of material in the members. Minimising $\Sigma R(l+2j)$ thus reduces to minimising ΣR. Making the sum of the bar forces as small as possible implies using a small number of bars. The minimum number of bars which can react the forces shown in Fig. 2 is three. They may be arranged as CD, CE and CF, of lengths l_1, l_2 and l_3, in which case the bar forces are P_1, P_2 and P_3, or as EF, FD and DE, of lengths s_1, s_2 and s_3, when we shall denote the bar forces by R_1, R_2 and R_3.

Considering the second arrangement, we take moments about F for the equilibrium of joint D, and obtain

$$R_3 q = P_1 p$$

where p and q are defined in Fig. 2.

Thus

$$R_3 l_1 = \frac{P_1 p l_1 s_3}{q s_3}$$

$$= P_1 a_2 s_3$$

where $a_2 A$ is the area of triangle CFD and A is the area of triangle DEF.

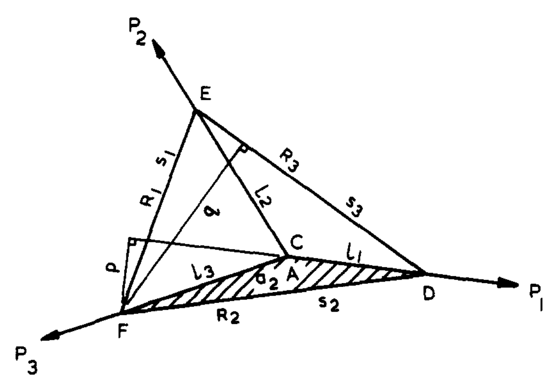

Fig. 2.

Similarly

$$R_3 l_2 = P_2 a_1 s_3.$$

Adding and dividing by $l_1 + l_2$,

$$R_3 = (P_1 a_2 + P_2 a_1)\frac{s_3}{l_1 + l_2}.$$

Summing for the three bar forces

$$R_1 + R_2 + R_3 = P_1\left(\frac{a_2 s_3}{l_1 + l_2} + \frac{a_3 s_2}{l_1 + l_3}\right) + P_2\left(\frac{a_3 s_1}{l_2 + l_3} + \frac{a_1 s_3}{l_2 + l_1}\right) + P_3\left(\frac{a_1 s_2}{l_3 + l_1} + \frac{a_2 s_1}{l_3 + l_2}\right).$$

Since $a_i + a_{i+1} < 1$ and $s_k < l_{k+1} + l_{k+2}$, where i and k are 1, 2 or 3, each of the factors of P_1, P_2 and P_3 is less than unity. It follows that

$$R_1 + R_2 + R_3 < P_1 + P_2 + P_3,$$

and the forces should be reacted by a triangular rather than a tripedal framework.

3. THE BEAM CARRYING A CENTRAL LOAD

3.1 *Full-space framework*

The general case of the singly symmetrical three-load system is discussed in Parkes[7]. The particular solution when the loads are parallel was first given by Michell[2]. The Michell optimum beam, where there is no restriction on the space which the structure can occupy, is shown in Fig. 3(a). The volume V of the bar material is $(\frac{1}{2} + \pi/4)Ps/\sigma$, so that $V\sigma/Ps$ is equal to 1·2854. The framework has an infinite number of joints, so that for non-zero values of j it cannot be the optimum form.

Fig. 3(b) shows a framework with five joints which is the optimum form for large values of j. The total volume of material is given by

$$V\sigma/Ps = \tfrac{1}{2}(\operatorname{cosec}\alpha \sec\alpha + \tan\alpha) + (j/s)(2\operatorname{cosec}\alpha + 4).$$

This is a minimum when

$$j/s = (2\tan^2\alpha - 1)/4\cos\alpha.$$

For $j = 0$, $\tan\alpha = 1/\sqrt{2}$ and $V\sigma/Ps$ is equal to $\sqrt{2}$. For other values of j/s the volume is plotted in Fig. 5 and the angle α in Fig. 7.

Figure 3(c) shows a framework with seven joints. The volume is given by

$$V\sigma/Ps = \tfrac{1}{2}\{\cot\beta + (\sin\alpha\operatorname{cosec}\beta + \sin\beta\operatorname{cosec}\alpha)\operatorname{cosec}(\alpha+\beta)\}$$
$$+(j/s)(\cot\alpha + \cot\beta + 2\operatorname{cosec}\alpha + 2\operatorname{cosec}\beta + 2).$$

Differentiating with respect to α and β we obtain two simultaneous equations for minimum

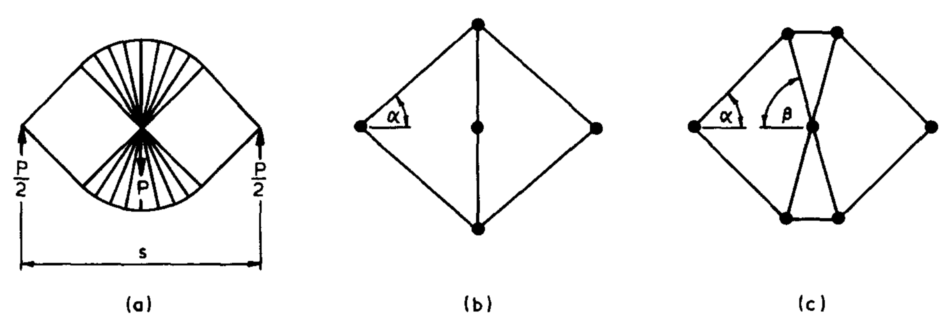

Fig. 3.

volume which can be re-phrased, with considerable algebraic manipulation, as

$$\tfrac{1}{2}(2\sin^2\alpha - \sin^2\beta) + (j/s)\{(2\cos\beta + 1)\sin^2\alpha - (2\cos\alpha + 1)\sin^2\beta\} = 0$$

and

$$\tfrac{1}{2}\{(2\sin^2\alpha - \sin^2\beta)\cos(\alpha+\beta) + \sin(\alpha+2\beta)\sin\beta\}\operatorname{cosec}(\alpha+\beta)$$
$$+(j/s)\{(2\cos\beta+1)\sin\alpha\cos\alpha + (2\cos\alpha+1)\sin\beta\cos\beta\} = 0.$$

For $j = 0$, $\alpha = \sin^{-1}(\sqrt{7}/4)$, $\beta = \sin^{-1}(\sqrt{14}/4)$ and $V\sigma/Ps = \sqrt{7}/2 = 1{\cdot}3229$. For other values of j, we eliminate j/s between the two preceding equations and obtain

$$\sin\alpha(1 + 4\cos\alpha - 2\cos\beta)\sin(\alpha+2\beta) + (1 + 2\cos\beta)(2\sin^2\alpha - \sin^2\beta) = 0.$$

This equation can be solved to provide corresponding pairs of values for α and β and these can be substituted in one of the preceding equations to find j/s and thence $V\sigma/Ps$. The volume is plotted in Fig. 5 and the values of α and β in Fig. 7.

On examining Fig. 5 it will be seen that for values of j/s less than about 0·093 the framework with seven joints has a smaller total volume than that with five joints. Frameworks with 9, 11, 13 .... joints could be devised, each approximating more closely to the Michell form, which would "round-in" the corner between the 7-joint and ∞-joint (Michell) lines in Fig. 5. The potential gain is, however, very small since the difference between the Michell and 7-joint volumes at $j = 0$ is less than 3 per cent. It seems likely that in the present case each of the frameworks with 5, 7, 9, 11, 13 .... joints would contribute to the final minimum volume polygon in Fig. 5. However, each member of a series of frameworks with ascending numbers of joints is not necessarily a minimum volume framework at some value of j, as will be seen when we consider the half-space beam (3.2).

Because of the convex form of the polygon of minimum volume, the penalty associated with the joint material increases less severely than a linear function of the joint radius. For $j/s = 0{\cdot}1$, the framework uses 67 per cent more material than the Michell form ($j/s = 0$). For $j/s = 0{\cdot}2$ there is 122 per cent of additional material.

3.2 *Half-space framework*

The Michell optimum beam, where the structure is required to lie on one side only of the line joining the supports, is shown in Fig. 4(a). $V\sigma/Ps$ is equal to $\pi/2$. The framework with the least number of joints which is capable of reacting the loads is shown in Fig. 4(b). It has four joints and its volume is given by

$$V\sigma/Ps = \operatorname{cosec}\alpha\sec\alpha + (j/s)(2\operatorname{cosec}\alpha + 2\cot\alpha + 4).$$

This is a minimum when

$$j/s = (\tan^2\alpha - 1)/2(1 + \cos\alpha).$$

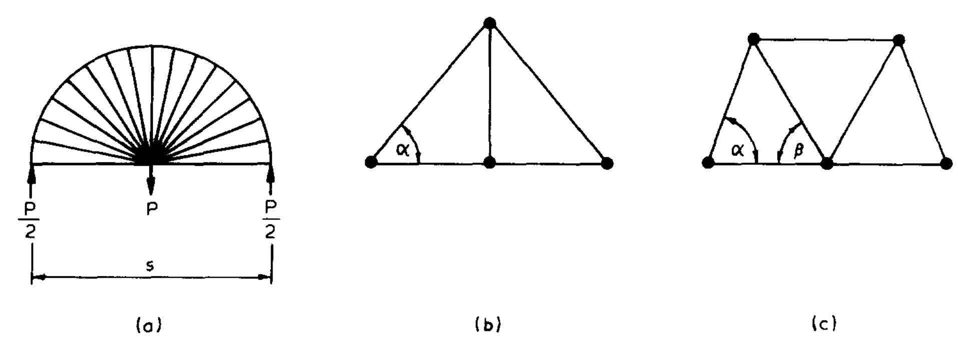

Fig. 4.

For $j = 0$, $\alpha = \pi/4$ and $V\sigma/Ps$ is equal to 2. For other values of j/s the volume is plotted in Fig. 6 and the angle α in Fig. 7.

Fig. 4(c) shows a five-joint framework lying entirely in the half-space above the line joining the supports. Its volume is given by

$$V\sigma/Ps = \tfrac{1}{2}\{\cot\alpha + \cot\beta + (\sin\alpha \operatorname{cosec}\beta + \sin\beta \operatorname{cosec}\alpha)\operatorname{cosec}(\alpha+\beta)\}$$
$$+ (j/s)(3\cot\alpha + \cot\beta + 2\operatorname{cosec}\alpha + 2\operatorname{cosec}\beta + 2).$$

Differentiating with respect to α and β we obtain two equations which can be re-combined to give

$$\sin^2\alpha - \sin^2\beta + (j/s)\{(2\cos\beta + 1)\sin^2\alpha - (2\cos\alpha + 3)\sin^2\beta\} = 0$$

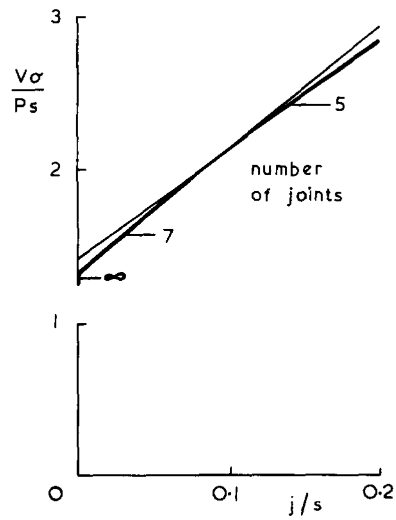

Fig. 5. Volumes of full-space optimum beams.

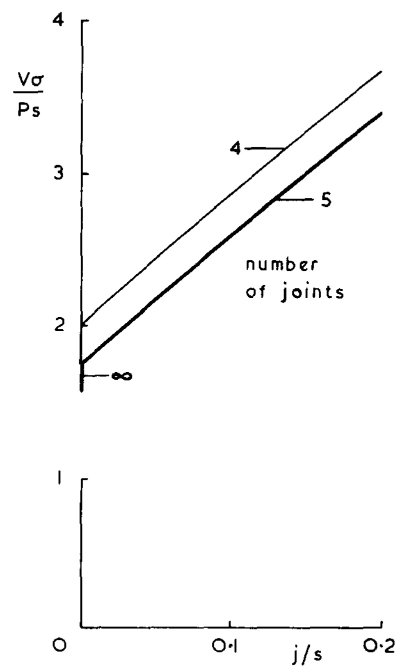

Fig. 6. Volumes of half-space optimum beams.

and

$$\tfrac{1}{2}\{(\sin^2\alpha+\sin^2\beta)\cot(\alpha+\beta)+\sin\alpha\cos\alpha+\sin\beta\cos\beta\}$$
$$+(j/s)\{(2\cos\beta+1)\sin\alpha\cos\alpha+(2\cos\alpha+3)\sin\beta\cos\beta\}=0.$$

When $j=0$, $\alpha=\beta=\pi/3$ and $V\sigma/Ps$ is equal to $\sqrt{3}$, some 10 per cent higher than the Michell value. For other values of j, we eliminate j/s and obtain

$$(2\cos\alpha+3)\sin\alpha\sin(\alpha+2\beta)-(2\cos\beta+1)\sin\beta\sin(2\alpha+\beta)=0.$$

This equation can be solved to provide corresponding pairs of values for α and β (in fact β is almost invariant at $\pi/3$ for a considerable range of α). These values of α and β are then substituted in one of the preceding equations to determine j/s and thence $V\sigma/Ps$. The volume is plotted in Fig. 6 and the angles α and β in Fig. 7.

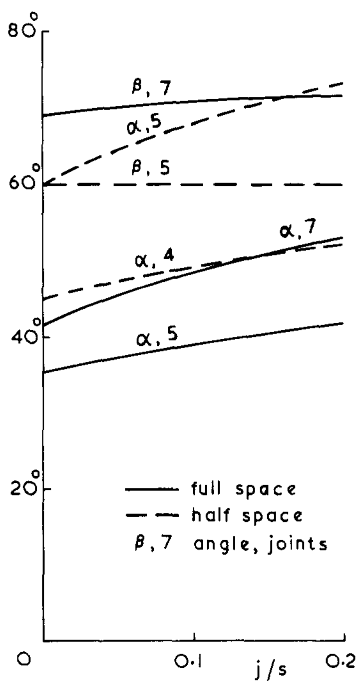

Fig. 7. Angles of optimum frameworks.

On examining Fig. 6 it will be seen that the volume of the five-joint framework is always less than that of the four-joint framework and so the line corresponding to this latter structure never forms part of the polygon of minimum volume. It would, of course, be possible to find frameworks having more than five joints to "round-in" the corner between the five-joint and ∞-joint (Michell) lines in Fig. 6.

For $j/s=0{\cdot}1$, the material consumption is 65 per cent greater than for the Michell optimum beam, and for $j/s=0{\cdot}2$ the excess is 116 per cent, values which are very similar to those for the full-space framework.

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